IB Diploma Mathematics AA HL Exploration

May 2021 exam session

# Maximizing the Spinning Time of a Top

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**Contents**

## *Section 1 : Introduction*

In the movie *Inception*, a spinning top is used by the protagonist to determine the reality they are in. As an avid fan of the movie, I bought myself a spinning top. The way the top stays upright for an extended period fascinated me. Whilst spinning the top, I noticed the shape of this top were different from the ones encountered in my childhood, which often have a circular base. It made me wonder – how does its shape affect the duration it spins for? What is the most efficient shape to maximize its spinning time ?

Figure 1 – Spinning top from the movie *Inception* [1]

When the opportunity arose, I was determined to satisfy my curiosity. Further background research has made me realised the complexity, and a lack of available information on this matter, making it even more exciting for me to carry out this investigation.

The aim of my exploration is to maximize the spinning time of a top through varying its base shape. To effectively investigate the influence of shape on spinning time, I will be keeping other variables, such as volume, density and initial angular velocity fixed. Moreover, to simplify the problem, I will assume that the torque exerted by air drag is independent of the shape of the top (but in reality, drag force depends on surface area and shape). This is to keep the influence of friction and air drag consistent, allowing me to solely focus on angular momentum. With the following simplifications in mind, I will first determine factors that affect the spinning time of the top.

## *Section 2 : Background Physics*

### *2.1 Forces on a Spinning Top*

To understand the motions of a spinning object, I must first understand torque $\vec{\tau}$ and angular momentum, $\vec{L}$. Torque, defined as $\vec{\tau} = \vec{r} \times \vec{F}$ (vectors quantities will be explained in *section 2.2*), is the result of applying a force, $\vec{F}$ to rotate an object around an axis, where $\vec{r}$ is the distance from the axisof rotation to the object (Figure 2 ) [2].

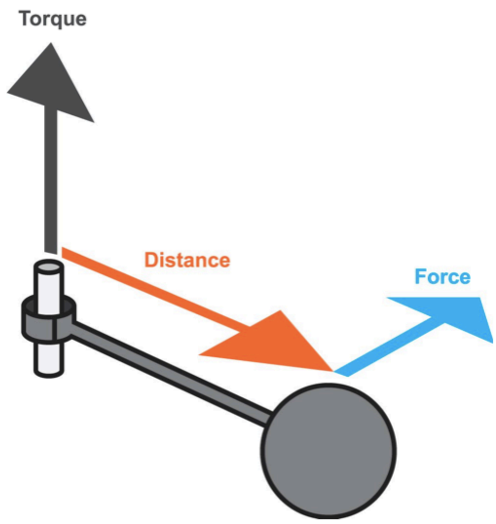


Figure 2– Forces exerted on a spinning top [2]

On the other hand, angular momentum is the rotational analog of linear momentum $\vec{p}$, defined as $\vec{L} = \vec{r} \times \vec{p}$. There are 2 special features to angular momentum : it is a conserved quantity, and that its rate of change is torque, $\vec{\tau} = \frac{d\vec{L}}{dt}$ [3]. This will be explored and discussed in more detail in *section 3.1*.

To determine factors influencing the spinning time of a top, I first examined the forces acting on it. Figure 3 shows an ideal cone-shaped top rotating about its axis of symmetry on a flat surface.

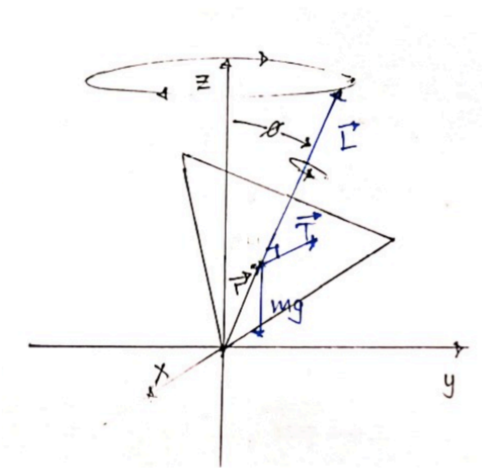


Figure 3 – Diagram of a spinning top

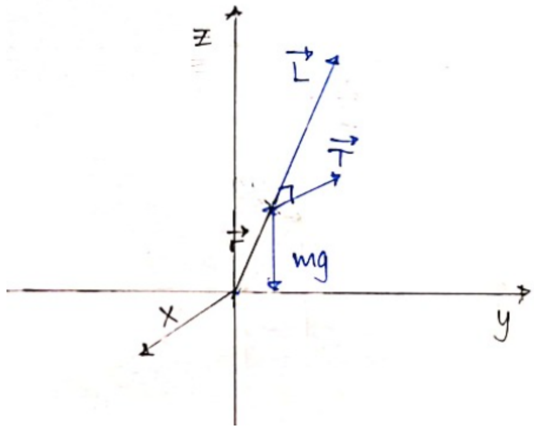


Figure 4 – Forces exerted on a spinning top

When a top is spinning, it has got an angular momentum $\vec{L}$ along the axis of the top. The force of gravity $m\vec{g}$ , where $m$ is the mass of the top and $\vec{g}$ is the gravitational acceleration, creates a torque $\vec{\tau}$ perpendicular to $\vec{L}$, which in turn changes the direction of the top's rotational axis (Figure 4).

When gravity is the only external force acting on the top, the magnitude of angular momentum is constant but it's direction changes. Hence the spinning top precesses in a constant precession angle $\phi$ provided that the angular speed is above a certain threshold. Below this threshold, the top starts to wobble and precession of the top becomes complicated (which will be discussed in *section 5.2*) [4]. Such complications will be ignored in my exploration.

However, in reality, air drag and friction from the ground creates an external torque that causes a deceleration. This slows the top down and reduces the magnitude of angular momentum. When the angular momentum drops below a certain threshold, the top no longer precesses steadily and gravity causes the top to eventually fall over.

To maximize the spinning time of a top, I would have to minimize the deceleration caused by air drag and friction. By assuming that the torque exerted by air drag is independent of the shape of the top (stated in *section 1*), this allows me to solely focus on the change of angular momentum. To do so, I first looked into the relationship between angular momentum and torque of a point mass.

## 2.2 Dot and Cross Product

This investigation will be revolved around vector quantities, hence it is important to understand how to manipulate vectors. As opposed to scalar quantities that only have a magnitude, vector quantities have both magnitude and direction. Vectors are denoted $\vec{a}$ and can be multiplied using dot product (e.g. $\vec{a} \cdot \vec{b}$) or cross product (e.g. $\vec{a} \times \vec{b}$).

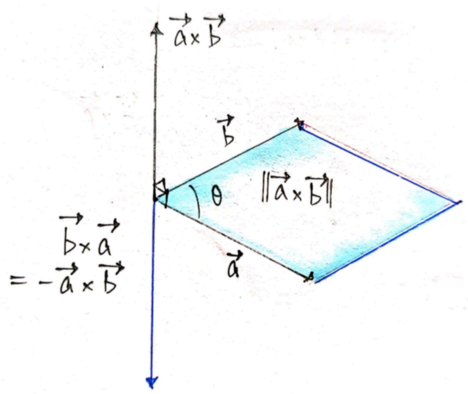


Figure 5 – Cross Product

Scaler product/ dot product

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| cos\theta$$

Vector product/ cross product

$$\vec{a} \times \vec{b} = \|\vec{a}\| \|\vec{b}\| sin\theta$$

where the magnitude of vector $\vec{a}$ is denoted by $\|\vec{a}\|$ [5]

The resultant vector from the cross product has a direction that is perpendicular to both $\vec{a}$ and $\vec{b}$, and has a magnitude equal to the area of a parallelogram with vectors $\vec{a}$ and $\vec{b}$ as sides [6]. The direction of the resultant vector can be determined by the right hand rule. It is important to note that the order in which the vector is written in determines the direction of the resultant vector, where $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (Figure 5)

### *Section 3 : Moment of Inertia*

#### *3.1 Torque and Angular Momentum of a Point Mass*

The angular momentum of an extended object is defined as the integral sum of its constituent particles [7]. That is, by adding up the angular momentum of particles that make up an object, I can find its angular momentum. To do so, I first considered the case of a point particle.

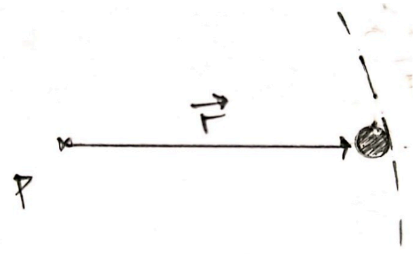

Figure 6 – Diagram of Point Particle

Let $\vec{r}$ be the displacement of the particle from a point $P$ and $\vec{p} = m\vec{v}$ be the linear momentum of the particle, where $m$ is the mass of the particle and $\vec{v}$ is its linear velocity (Figure 6). As angular momentum is defined as $\vec{L} = \vec{r}\times\vec{p}$ (*section 2.1*) then the angular momentum about point $P$ is would be

$$\vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} \text{ [8]}$$

It is important to note that the magnitude of $\vec{L}$ depends on the choice of reference point $P$.

If there is a force $\vec{F}$ acting on the particle, then according to Newton's second law $\vec{F} = \frac{d\vec{p}}{dt}$, there will be a change in momentum, $\vec{p}$. This would result in a torque, $\vec{\tau}$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\rightarrow \vec{r}\times\vec{F} = \vec{r}\times\frac{d\vec{p}}{dt}$$

$$\vec{\tau} = \vec{r}\times\frac{d\vec{p}}{dt}$$

As angular momentum, $\vec{L} = \vec{r}\times\vec{p}$ , I can write torque in terms of angular momentum

To do so, I used the product rule for cross product, $\frac{d(\vec{r}\times\vec{p})}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt}\times\vec{p}$ [9]. Thus

$$\vec{\tau} = \vec{r}\times\frac{d\vec{p}}{dt} = \frac{d(\vec{r}\times\vec{p})}{dt} - \frac{d\vec{r}}{dt}\times\vec{p} = \frac{d(\vec{r}\times\vec{p})}{dt} - \vec{v}\times(m\vec{v}) = \frac{d\vec{L}}{dt}$$

Here, I have deduced that torque is proportional to the rate of change of angular momentum. This shows that without external torque, angular momentum is conserved. However, understanding this relationship alone is not sufficient to determine how do I minimize deceleration by external torques (such as air drag and friction). Thus, my next step will be establishing the relationship between torque, angular momentum and angular acceleration.

### *3.2 From Angular Momentum to Angular Acceleration*

If the particle is undergoing any sort of circular motion (doesn't necessarily have to be uniform), then I can rewrite linear velocity $\vec{v}$ in terms of angular velocity $\vec{\omega}$, defined via $\vec{v} = \vec{\omega}\times\vec{r}$

$$\vec{L} = m\vec{r}\times(\vec{\omega}\times\vec{r}) = mr^2\vec{\omega}$$

From this, I have established that $\vec{L} \propto \vec{\omega}$. The proportionality constant is called the *moment of inertia* of the particle with respect to the point *P*, which is conventionally denoted by $I$, thus $\vec{L} = I\vec{\omega}$.

In the case of a point particle, $I = m\vec{r}^2$. The moment of inertia will differ depending on the mass distribution of the object, but $\vec{L} \propto \vec{\omega}$ is general [11]. With this, I can rewrite torque acting on the point particle in terms of its moment of inertia.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = mr^2\frac{d\vec{\omega}}{dt} = I\frac{d\vec{\omega}}{dt}$$

To simplify the relationship, I rewrote angular velocity in terms of angular acceleration, $\vec{\alpha}$, where $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$. Hence

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$$

The relationship $\vec{\tau} = I\vec{\alpha}$ is generally true provided that the moment of inertia remains constant [12]. From $\vec{\tau} = I\vec{\alpha}$, I realised that the greater the moment of inertia, the smaller the angular deceleration caused by air drag and friction. This is based on the earlier assumption that torque exerted by air drag is independent of the shape of the top. By assuming this, the product of $I\vec{\alpha}$ remains a fixed value. Thus, the spinning time of a top will be maximized when the moment of inertia is maximized.

### 3.3 Moment of Inertia of a Circular Disc

As mentioned in *section 3.1*, the angular momentum of an extended object is defined as the integral sum of its constituent particles. Due to keeping the initial angular velocity constant, the angular momentum of a spinning top is directly proportional to its moment of inertia ($\vec{L} = I\vec{\omega}$). Hence I can calculate the moment of inertia of an object by adding up the moment of inertia of its constituent particles.To simplify calculations, I will consider a spinning top as a stack of circular discs, where its moment of inertia will be the sum of the moment of inertia of the discs.

Suppose there is a circular disk with area $A$, thickness $\Delta h$, radius $R$ and mass $m$. To calculate the moment of inertia of the disc, I broke it down into $n$ small fragments and treated each fragment as point particles (Figure 7). For the $i$th fragment, its moment of inertia would be

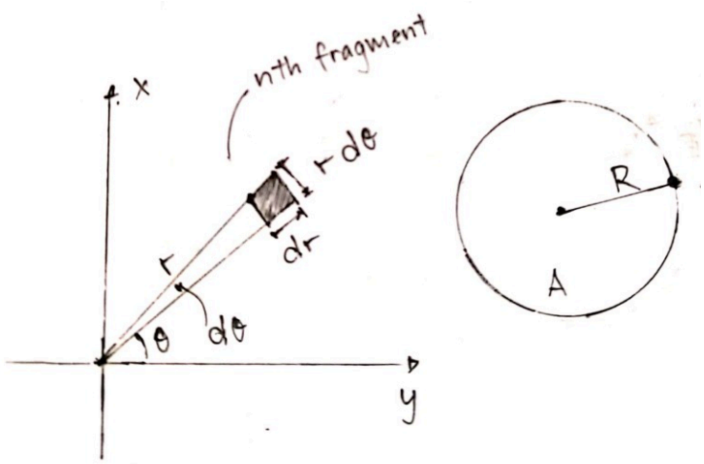


Figure 7 – Diagram of Fragment and Disc

$$\Delta I_i \cong r_i^2 \Delta m_i$$

Where $r_i$ is the distance from the axis of rotation to the $i$th fragment. Hence the sum of all particles would be

$$I_d = \sum_{i=1}^{n} \Delta I_i \cong \sum_{i=1}^{n} r_i^2 \Delta m_i$$

Where $I_d$ is the moment of inertia of a disc. At this point, I encountered complications in integrating $\Delta m_i$. This is because $\Delta m_i$ is dependent upon the location of the fragment (which makes it a function of $r_i$. To proceed further, I rewrote $\Delta m_i$ in terms of $r_i$ . To do so, I first defined the density of the material as $\rho = \frac{m}{V} = \frac{m}{Ah}$ where $V$ is its volume [ 13]. Rearranging and subsituting for $\Delta m_i$

$$\Delta m_i = \rho \Delta V_i = \rho\, \Delta h\, \Delta A_i = \rho \Delta h \cdot \Delta r \cdot r_i \Delta \theta$$

Then from Figure 7, I realised that the area of a fragment, $\Delta A_i$ can be obtained through multiplying its sides, thus $\Delta A_i = \Delta r \cdot r_i \Delta \theta$ . With this, I can rewrite the equation as

$$\Delta m_i = \rho\, \Delta h\, \Delta A_i = \rho \Delta h \cdot \Delta r \cdot r_i \Delta \theta$$

Subsituting this into the formula for moment of inertia

$$I_d \cong \sum_{i=1}^{n} r_i^2 \cdot \rho \Delta h \cdot \Delta r \cdot r_i \Delta \theta$$

By taking the limit $n \to \infty$, the summation becomes an integral

$$I_d = \int_{disc} r^3 \cdot \rho \Delta h \cdot\, dr\, d\theta$$

where $\int_{disc}$ is an abbreviation for integration over the whole disc. As the disc is circular with a radius of $R$, the $r$-intergral is from 0 to $R$ and the $\theta$ integral is from 0 to $2\pi$. Therefore the integral would be

$$\int_0^{2\pi} \left( \int_0^R r^3 \cdot \rho \Delta h \cdot\, dr \right) d\theta$$

Taking out the constant, $\rho \Delta h$

$$= \rho \Delta h \int_0^{2\pi} \left( \int_0^R r^3 \, dr \right) d\theta$$

Integrating and substituting the boundaries of $\int_0^R r^3 \, dr$

$$= \rho \Delta h \int_0^{2\pi} \left[ \frac{r^4}{4} \right]_0^R \, d\theta$$

$$= \rho \Delta h \int_0^{2\pi} \frac{R^4}{4} d\theta$$

Finally integrating and subsituting the boundaries of $\int_0^{2\pi} \frac{R^4}{4} d\theta$

$$= \frac{1}{4} \rho R^4 \Delta h [\theta]_0^{2\pi}$$

$$= \frac{1}{4} \rho R^4 \Delta h (2\pi)$$

$$I_d = \frac{1}{2} \rho \pi R^4 \Delta h$$

With the moment of inertia of a single disc, I can then proceed further by using this to calculate the moment of inertia of a top.

### 3.3 Moment of Inertia of a Top

As mentioned earlier, a cylindrically symmetric body, such as a top, can be considered as stacked discs, where its moment of inertia will be the sum of the moment of inertia of the discs.

Figure 8 shows such a disc with height $H$, spilt into $N$ discs where each of them has a thickness $\Delta h$. The $i$-th disc at height $h_i$ has a radius (as a function of height) $r_i = r(h_i)$. From what I have established above, the moment of inertia for one disc, $\Delta I_i$ would then be

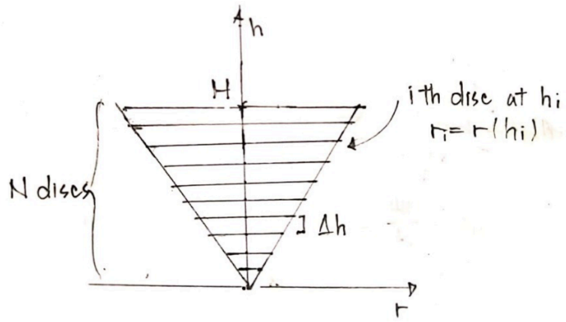


Figure 8 – Stacked Discs

$$\Delta I_i = \frac{1}{2}\rho\pi r_i^4 \Delta h$$

Subsituding $r_i = r(h_i)$

$$\Delta I_i = \frac{1}{2}\rho\pi r_i^4 \Delta h = \frac{1}{2}\rho\pi \cdot r(h_i)^4 \Delta h$$

Hence the moment of inertia of the top, as the sum of the moment of inertia of its respective discs would be

$$I_{top} = \sum_{i=1}^{N} \Delta I_i = \sum_{i=1}^{N} \frac{1}{2}\rho\pi \cdot r(h_i)^4 \, \Delta h$$

Taking the limit $N \to \infty$

$$I_{top} = \frac{1}{2}\rho\pi \int_0^H r(h)^4 \, dh$$

Where the $h$-integral is from 0 to $H$ due to that being the height of the top.

## *Section 4 : Optimization*

### *4.1 Varying the Shape of a Top*

Now that I have the formula for moment of inertia of a spinning top, I wanted to investigate the effect changing the base shape has on its moment of inertia. To do so, I set the slope of the top as a function of $h$ and investigated the moment of inertia for various shapes.

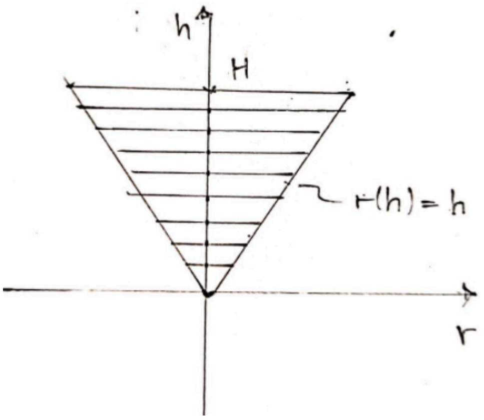


Figure 9 – top with sides $r(h) = h$

I started off by first considering the most basic top shape – a cone (Figure 9). By substituting $r(h) = h$ and integrating the function, the moment of inertia becomes

$$I = \frac{1}{2}\rho\pi\int_0^H h^4\,dh = \frac{1}{2}\rho\pi\left[\frac{1}{5}h^5\right]_0^H = \frac{1}{10}\rho\pi H^5$$

Substituting the volume of a cone, $V = \frac{1}{3}\pi H^3$ into the moment of inertia of the top

$$I = \frac{1}{10}\rho\pi H^5 = \frac{3}{10}\rho V H^2$$

And from $m = \rho V$

$$I = \frac{3}{10}\rho V H^2 = \frac{3}{10}mH^2$$

Afterwards, I investigated the moment of inertia of tops with sides of various power coefficient of $h$. To enable a better comparison between the moment of inertia of various shapes, I fixed the volume of the tops as V= $\frac{\pi H^3}{3}$ (could be any consistent value, this is just for convenience).

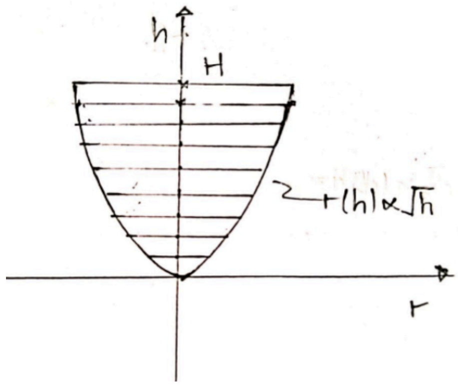


Figure 10 – top with sides $r(h) \propto \sqrt{h}$

The volume of a top with sides $r(h) = k\sqrt{h}$ (Figure 10) can be written as the integral sum of its respective discs. As such a disc would have a volume of $\pi r(h_i)^2$, the volume of such a top would be

$$V \cong \sum_{i=1}^{N} \pi r(h_i)^2 \, \Delta h$$

Taking the limit $N \to \infty$

$$V = \int_0^H \pi r(h)^2 dh$$

The proportionality constant $k$ of a top with volume $V$ and sides $r(h) = k\sqrt{h}$ (Figure 10) could be obtained through substituting $r(h) = k\sqrt{h}$ into its volume, where V= $\frac{\pi H^3}{3}$

$$\frac{\pi H^3}{3} = \int_0^H \pi (k\sqrt{h})^2 \, dh$$

Taking out the constant, $\pi k^2$

$$\frac{\pi H^3}{3} = \pi k^2 \int_0^H (\sqrt{h})^2 \, dh$$

Integrating and substituting the boundaries of $\int_0^H (\sqrt{h})^2 \, dh$

$$\frac{\pi H^3}{3} = \pi k^2 \left[\frac{1}{2} h^2\right]_0^H$$

$$\frac{\pi H^3}{3} = \pi k^2 \frac{1}{2} H^2$$

Solving for $k$

$$k^2 = \frac{2}{3} H$$

$$k = \sqrt{\frac{2H}{3}}$$

Substituting into $r(h)$

$$r(h) = \sqrt{\frac{2Hh}{3}}$$

With this, I can subsitute $r(h) = \sqrt{\frac{2Hh}{3}}$ into the formula to obtain its moment of inertia

$$I = \frac{1}{2}\rho\pi \int_0^H \left(\sqrt{\frac{2Hh}{3}}\right)^4 dh$$

$$= \frac{1}{2}\rho\pi \int_0^H \frac{4}{9}h^2 H^2 \, dh$$

$$= \frac{4}{18}\rho\pi H^2 \int_0^H h^2 \, dh$$

Integrating and substituting

$$= \frac{4}{18}\rho\pi H^2 \left[\frac{1}{3}h^3\right]_0^H$$

$$= \frac{4}{18}\rho\pi H^2 \cdot \frac{1}{3}H^3$$

Substituting V= $\frac{\pi H^3}{3}$ and from $m = \rho V$

$$I = \frac{2}{9}\rho V \, H^2 = \frac{2}{9}m \, H^2$$

From this, I realised that the moment of inertia of a top with sides $r(h) \propto \sqrt{h}$ is smaller than that of a cone-shaped top with sides $r(h) = h$. Thus, I hypothesized that the sharper the base of the top, the larger its moment of inertia.

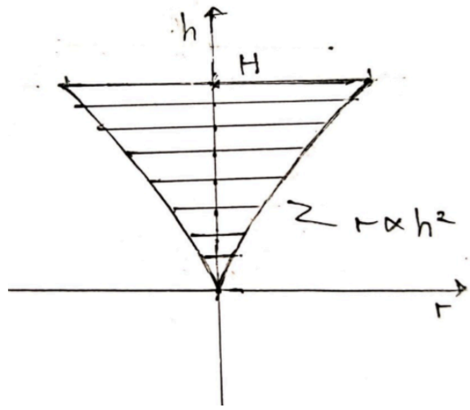


Figure 11 – top with sides $r(h) \propto h^2$

To investigate this, I considered the moment of inertia of a top with sides $r(h) \propto h^2$, which has a sharper base than the 2 cases I have examined above (Figure 11). With $V = \frac{\pi H^3}{3}$, the proportionality constant $k$ would be

$$V = \frac{\pi H^3}{3} = \int_0^H \pi \cdot k^2 h^4 \, dh$$

$$k = \sqrt{\frac{5}{3}\frac{1}{H}}$$

Hence its moment of inertia would be

$$I = \frac{1}{2}\rho\pi \int_0^H \left(kh^2\right)^4 dh = k^4 \cdot \frac{1}{18}\rho\pi \cdot H^9 = \frac{25}{54}\rho\pi H^5 = \frac{25}{54} m H^2$$

As the moment of inertia of a top with sides $r(h) \propto h^2$ appears to have the highest moment of inertia among the cases I have examined so far, this supports my hypothesis that the sharper the base of the top, the larger its moment of inertia.

## 4.2 Optimizing the Shape of a Top

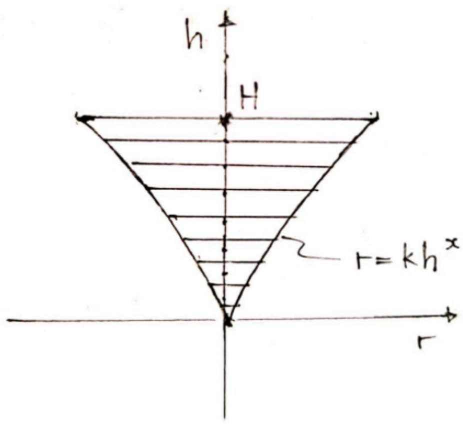


Figure 12 – top with sides $r(h) \propto h^x$

From the cases I examined so far, it appears to be a general trend where the larger the power coefficient of $h$, the larger its moment of inertia. This raises a couple of questions : is there a power coefficient of $h$ in which the moment of inertia is maximized ? If so, what is it ? To find out, I considered a top with sides $r(h) = kh^x$ (Figure 12) and volume $V = \frac{\pi H^3}{3}$.

The proportionality constant $k$ of such a top would be

$$V = \frac{\pi H^3}{3} = \int_0^H \pi (k h^x)^2 \, dh$$

Taking out the constant, $\pi k^2$ and simplifying

$$\frac{H^3}{3} = k^2 \int_0^H h^{2x} \, dh$$

Integrating and substituting the boundaries of $\int_0^H h^{2x} \, dh$

$$\frac{H^3}{3} = k^2 \left[ \frac{1}{2x+1} h^{2x+1} \right]_0^H$$

$$\frac{H^3}{3} = \frac{k^2}{2x+1} \; H^{2x+1}$$

Rearranging and simplifying

$$\frac{k^2}{2x+1} \; H^{2(x-1)} = \frac{1}{3}$$

Solving for $k$

$$k^2 = \frac{2x+1}{3H^{2(x-1)}}$$

$$k(x) = \pm \sqrt{\frac{2x+1}{3}} \, \frac{1}{H^{x-1}}$$

As the top is cylindrically symmetric, it doesn't matter whether I take the positive or negative value of $k(x)$. Still, I will take the positive value of $k(x)$ so that *r* is positive.

Substituting $k(x)$ into formula for the moment of inertia, $I = \frac{1}{2} \rho \pi \int_0^H r(h)^4 \, dh$ and integrating with respect to $dh$

$$I(x) = \tfrac{1}{2} \rho \pi \textstyle\int_0^H (k(x) \, h^x)^4 \, dh$$

$$= \frac{1}{2} \rho \pi \int_0^H k(x)^4 \, h^{4x} \, dh$$

$$I(x) = \frac{1}{2}\rho\pi \cdot k(x)^4 \left[\frac{1}{4x+1}\ H^{4x+1}\right]$$

This gives me the moment of inertia for a a top with sides $r(h) = kh^x$ and volume $V = \frac{\pi H^3}{3}$.

To find the value of $x$ in which the moment of inertia is maximized, I will take the derivative of $I(x)$ and set it to zero, calculating $\frac{dI(x)}{dx} = 0$. Afterwards, I will plot the function to determine whether it is a maximum, minimum or point of inflexion.

$$\frac{dI(x)}{dx} = \frac{1}{2}\rho\pi \cdot \frac{d}{dx}\left(k(x)^4 \left[\frac{1}{4x+1}\ H^{4x+1}\right]\right) = 0$$

Although the derivative seems complicated at first glance, it can be tackled through a series of chain, product and quotient rules. Let $u_I$ be $k(x)^4$ and $v_I$ be $\frac{1}{4x+1}\ H^{4x+1}$, thus

$$\frac{d}{dx}\left(k(x)^4 \left[\frac{1}{4x+1}\ H^{4x+1}\right]\right) = \frac{d}{dx}u_I v_I = u_I \frac{dv_I}{dx} + v_I \frac{du_I}{dx}$$

First, I used chain rule to find $\frac{du_I}{dx}$

$$\frac{du_I}{dx} = \frac{d}{dx}(k(x)^4) = 4k(x)^3 \frac{dk(x)}{dx}$$

Afterwards, I used product rule to find $\frac{dv_I}{dx}$

$$\frac{dv_I}{dx} = \frac{d}{dx}\left(\frac{1}{4x+1}\ H^{4x+1}\right) = -\frac{4}{4x+1} \cdot H^{4x+1} \cdot \ln H - \frac{4}{(4x+1)^2} \cdot H^{4x+1}$$

Here, I used a series of chain rules to find $\frac{d}{dx}\left(\frac{1}{4x+1}\right)$ and $\frac{d}{dx}(H^{4x+1})$

Subsituting $\frac{du_I}{dx}$ and $\frac{dv_I}{dx}$ into $\frac{d}{dx}u_I v_I$

$$\frac{d}{dx}u_I v_I = u_I \frac{dv_I}{dx} + v_I \frac{du_I}{dx}$$

$$= k(x)^4 \cdot \left[-\frac{4}{4x+1} \cdot H^{4x+1} \cdot \ln H - \frac{4}{(4x+1)^2} \cdot H^{4x+1}\right] + 4k(x)^3 \frac{dk(x)}{dx} \cdot \left[\frac{1}{4x+1}\ H^{4x+1}\right]$$

Rearranging and simplifying

$$\frac{d}{dx}\left(k(x)^4\left[\frac{1}{4x+1}\ H^{4x+1}\right]\right) = k(x)^3\frac{4}{4x+1}H^{4x+1}\left[k(x)\left(\ln H - \frac{1}{4x+1}\right) + \frac{dk(x)}{dx}\right]$$

Subsituting back into $\frac{dI(x)}{dx}$

$$\frac{dI(x)}{dx} = \frac{1}{2}\rho\pi\cdot k(x)^3\frac{4}{4x+1}H^{4x+1}\left[k(x)\left(\ln H - \frac{1}{4x+1}\right) + \frac{dk(x)}{dx}\right]$$

Solving for $\frac{dI(x)}{dx} = 0$

$$\frac{1}{2}\rho\pi\cdot k(x)^3\frac{4}{4x+1}\ H^{4x+1}\left[\left(\ln(H) - \frac{1}{4x+1}\right)k(x) + \frac{dk(x)}{d(x)}\right] = 0$$

To proceed further from this point, I found the derivative of $k(x)$ via $\frac{dk(x)}{dx} = \frac{d}{dx}\ k(x)$.

Subtituding $k(x) = \sqrt{\frac{2x+1}{3}}\ \frac{1}{H^{x-1}}$

$$\frac{dk(x)}{d(x)} = \frac{d}{dx}\left(\sqrt{\frac{2x+1}{3}}\right)\frac{1}{H^{x-1}}$$

Here, I used the product rule. Let $u_k$ be $\sqrt{\frac{2x+1}{3}}$ and $v_k$ be $\frac{1}{H^{x-1}}$, thus

$$\frac{d}{dx}\left(\sqrt{\frac{2x+1}{3}}\right)\frac{1}{H^{x-1}} = \frac{d}{dx}u_k v_k = u_k\frac{dv_k}{dx} + v_k\frac{du_k}{dx}$$

Using chain rule,

$$\frac{du_k}{dx} = \frac{d}{dx}\left(\sqrt{\frac{2x+1}{3}}\right) = \frac{3}{4\left(\frac{2x+1}{3}\right)^{\frac{1}{2}}} = \frac{1}{3}\left(\frac{2x+1}{3}\right)^{-\frac{1}{2}}$$

Using quotient rule,

$$\frac{dv_k}{dx} = \frac{d}{dx}\left(\frac{1}{H^{x-1}}\right) = -H^{-x+1}\cdot\ln H$$

Subsituting $\frac{du_k}{dx}$ and $\frac{dv_k}{dx}$ into $\frac{d}{dx}u_k v_k$

$$\frac{d}{dx}u_k v_k = u_k\frac{dv_k}{dx} + v_k\frac{du_k}{dx}$$

$$= -\left(\frac{2x+1}{3}\right)^{\frac{1}{2}} \cdot \frac{1}{H^{x-1}}\ln H + \frac{1}{3}\left(\frac{2x+1}{3}\right)^{-\frac{1}{2}} \cdot \frac{1}{H^{x-1}}$$

Rearranging and simplifying

$$= \frac{1}{H^{x-1}}\left(\frac{2x+1}{3}\right)^{\frac{1}{2}}\left(\frac{1}{2x+1} - \ln H\right)$$

Subtituding $k(x) = \sqrt{\frac{2x+1}{3}}\,\frac{1}{H^{x-1}}$

$$\frac{dk(x)}{d(x)} = k(x)\left[\frac{1}{2x+1} - \ln H\right]$$

Subsituding $\frac{dk(x)}{d(x)}$ back into $\frac{dI(x)}{d(x)}$

$$\frac{dI(x)}{d(x)} = \frac{1}{2}\rho\pi \cdot k(x)^3\frac{4}{4x+1}\ H^{4x+1}\left[\left(\ln H - \frac{1}{4x+1}\right)k(x) + \left(\frac{1}{2x+1} - \ln H\right)k(x)\right] = 0$$

Taking out the constant, $k(x)$

$$\frac{1}{2}\rho\pi \cdot k(x)^3\frac{4}{4x+1}\ H^{4x+1}k(x)\left[\left(\ln H - \frac{1}{4x+1}\right) + \left(\frac{1}{2x+1} - \ln H\right)\right] = 0$$

Dividing both sides by $\frac{1}{2}\rho\pi \cdot k(x)^3\frac{4}{4x+1}\ H^{4x+1}k(x)$

$$\ln H - \frac{1}{4x+1} + \frac{1}{2x+1} - \ln H = 0$$

$$= \frac{2x}{(4x+1)(2x+1)} = 0$$

$$x = 0$$

However $\frac{dI(x)}{dx} = 0$ doesn't indicate the nature of the function – this could be a maximum, minimum or point of inflexion. As calculating the 2nd derivative would be unefficient, I decided to plot the anti-derivative of $\frac{dI(x)}{dx}$ , which is $I(x)$, on a graph. This will enable me to examine the nature of this point.

$$I(x) = \frac{1}{2}\rho\pi \cdot k(x)^4 \left[\frac{1}{4x+1}\ H^{4x+1}\right]$$

Subsituding $k(x) = \sqrt{\frac{2x+1}{3}}\ \frac{1}{H^{x-1}}$

$$I(x) = \frac{1}{2}\rho\pi \cdot \left[\frac{(2x+1)^2}{9(4x+1)}\right] \cdot H^5$$

In order to ensure that the function does not have a dimension, I divided both sides by $\rho H^5$

Plotting $\frac{I(x)}{\rho H^5} = \frac{1}{2}\pi \cdot \left[\frac{(2x+1)^2}{9(4x+1)}\right]$

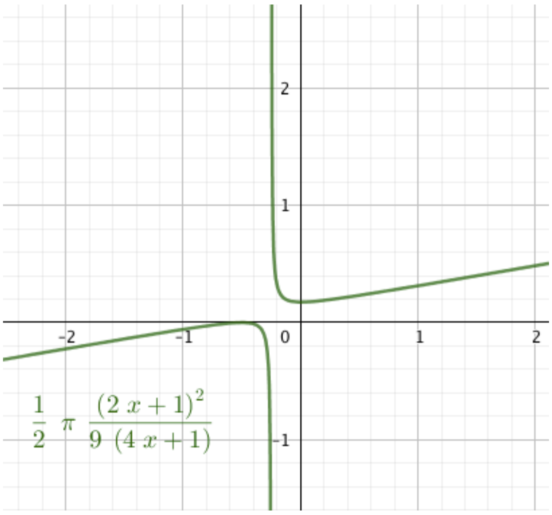


Figure 13 – Graph of $\frac{I(x)}{\rho H^5}$

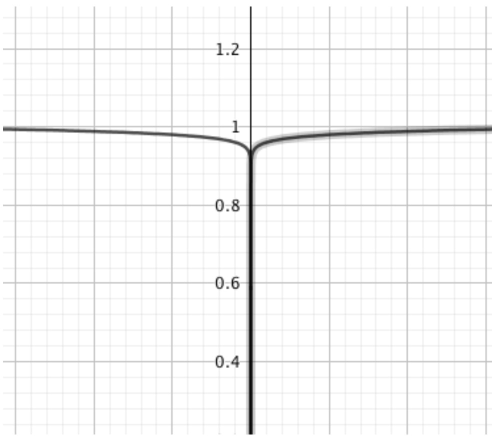


Figure 14 – Graph of $r(h) = h^{80}$

The graph shows that when $x = 0$, $\frac{dI(x)}{dx}$ is a minimum. As $I < 0$ is not physically possible, values of $I < 0$ will be ignored. From its derivative, I know that there is no maximum, hence as $x$ tends to infinity, moment of inertia $I$ tends to infinity.

This is in accordance with my hypothesis, where I proposed that the sharper the base of the top, the larger its moment of inertia. However, it is physically impractical to have a top with sides $r(h) \propto h^x$ where $x$ is a really large number (demonstrated in Figure 14). This is because such a top would have easily wobble over and could not maintain a steady state due to the fundamental laws of physics.

For instance, the top would easily come into contact with the ground when precession occurs, which immediate stops the top from spinning. A possible alternative solution and area of further investigation would be to impose physical constraints to the shape of the top (Figure 16). By limiting the angle between the base of the top and its maximum radius for example, this could decrease the chances of the top falling over due to the sides being in contact with the ground. M

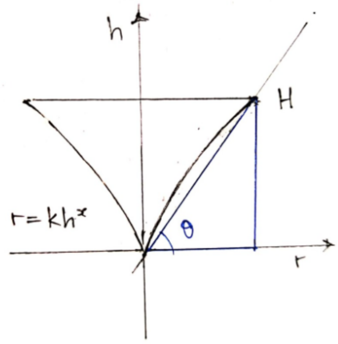


Figure 16 – Possible physical constrains to the shape of the top

## Section 5 : Conclusion

### 5.1 Conclusion

The aim of my exploration is to maximize the spinning time of a top through varying its base shape. To do so, I first analyzed forces acting on a spinning top. This has enabled me to identify factors that affect the spinning time of the top. Through my analysis, I realised that to maximize the spinning time of a top, I would have to minimize the deceleration caused by air drag and friction. To do so, I first looked into the relationship between angular momentum and torque of a point mass. Then, I used this knowledge to derive the relationship between angular acceleration and angular momentum. With this, I established that $\vec{L} = I\vec{\omega}$. As initial angular velocity is kept constant in my exploration, the spinning time of a top will be maximized when the moment of inertia is maximized. This is because the larger the moment of inertia, the smaller the angular deceleration caused by air drag and friction thus the longer the duration span.

With the new focus of my exploration on maximizing the moment of inertia of a top through varying its base shape, I looked into how could I calculate the moment of inertia of a spinning top. To do so, I first started by considering the case of a point particle, before moving on to circular discs. By considering a spinning top as a stack of circular discs, I have successfully derived an equation for the moment of inertia of a top based on its slope.

Afterwards, I applied my equation to tops of various shapes, including a cone-shaped top, a top with a circular base and a top with a thin base. From my calculations, I have realised that the top with a thin base had the largest moment of inertia. Thus, I hypothesized that the sharper the base of the top, the larger its moment of inertia.

To test my hypothesis, I considered a top with sides $r(h) = kh^x$ and volume $V = \frac{\pi H^3}{3}$. To find the value of $x$ in which the moment of inertia is maximized, I calculated the value of $x$ in which $\frac{dI(x)}{dx} = 0$ . Through a complex series of algebraic manipulations, I arrived at a solution. However $\frac{dI(x)}{d(x)} = 0$ doesn't indicate the nature of the function – this could be a maximum, minimum or point of inflexion. As calculating the 2$^{nd}$ derivative would be unefficient, I plotted the anti-derivative of $\frac{dI(x)}{d(x)}$ (which is $I(x)$) on a graph to examine the nature of this point. My graph of $I(x)$ reveals that there is no maximum to the function, thus as *x* tends to infinity, moment of inertia *I* tends to infinity.

Relating this to the aim of my exploration, I realised that it is physically impartical to have a top with sides $r(h) \propto h^x$ where *x* is a really large number. This is due to a number of physical limitations, outlined in *section 4.2*. However, additional criterias could be imposed to create a spinning top with the maximum moment of inertia. Such areas of further investigations will be explored in *section 5.3*.

## 5.2 Evaluation of Findings : Assumptions and Limitations

There are multiple assumptions that I made, which would have affected the validity of my findings. First, I assumed the torque exerted by air drag is independent of the shape of the top. In reality however, drag force depends on both surface area and shape of the top. This was done to simplify the problem, which enabled me to focus on how the shape of the top affects its moment of inertia, hence time span. Moreover, I made an implicit assumption of the tops having the same maximum height $H$ when comparing between tops of different shapes. This was done to allow effective comparison, but if time allows, the influence of varying the height $H$ of the top should be investigated. Furthermore, I only invested tops with slopes that obey the function, $r(h) \propto h^x$. This is mainly to simplify my exploration, and to allow my exploration to remain focused on a goal. Further areas of research could include slopes of other functions, such as sinusoidal functions and piecewise functions. Additionally, I only investigated the base of the top, disregarding the handles. Although this limits how generalizable my findings are, its effects could be easily calculated by adding the moment of inertia of the handle into the moment of inertia of the whole top. Lastly, I ignored the complications in precession when angular speed of a top drops below the threshold required to remain a steady precession. This is because at that point, the top starts to wobble and the precession becomes complicated (beyond my understanding of physics). However, it is still a general case where the lower the deceleration, the longer the top spins for. This is also a reason why it will be challenging to calculate the exact duration the top spins for. Realising such complications, I decided to focus my investigation on the relationship between spinning time of the top and its base shape as opposed to determining an exact duration.

## *Bibliography*


[1] (image) http://thegadgetflow.com/portfolio/inception-spinning-top-replica-15/

[2] (image) https://www.allaboutcircuits.com/tools/torque-conversion-calculator/

[3] [7] [8] [10] [11] [12] http://hyperphysics.phy-astr.gsu.edu/hbase/amom.html#am

[4] https://aapt.scitation.org/doi/abs/10.1119/1.4776195

[5] https://www.math.ucla.edu/~josephbreen/Understanding_the_Dot_Product_and_the_Cross_Product.pd

[6] https://ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/1.-vectors-and-matrices/part-c-parametric-equations-for-curves/session-19-velocity-and-acceleration/MIT18_02SC_pb_20_comb.pdf

[9] https://www.mathsisfun.com/algebra/vectors-cross-product.html

[13] https://www.thoughtco.com/what-is-density-definition-and-calculation-2698950