Mathematics Extended Essay:
Vibrating Strings and Musical Theory of Harmonics

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Research Question: To what extent do the motion of strings within stringed instruments described by Fourier Series give insight into consonance and dissonance in music theory?

Word Count: 3980

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1 Introduction

The inherent beauty and understanding that arises within the study of music can be quite evidently compared and connected to that of mathematical theory. Whether it be a guitar, violin, or piano, the sounds these instruments make have in common their origin, strings. By keeping a string taught between two points, one can produce a family of various sounds which can be used in union to compose the form of a pleasant melody [1]. It is the aim of this paper to analyse the sounds produced by these strings by their motion, and see how, by the interactions of these sounds, pleasant and unpleasant, i.e. consonant, and dissonant sounds can be produced. This is done by deriving the wave equation [2]; a solution is given by a method of separating variables, which the uniqueness [3] of that solution is then shown. From this then it is concluded how sounds described by the wave equation can produce consonant and or dissonant sounds. To focus this research, the following research question is necessary: *To what extent do the motion of strings within stringed instruments described by Fourier Series give insight into consonance and dissonance in music theory?*

2 Deriving the Wave Equation

Imagine a rope, tied at one end unto a pole or rod of sorts. The rope is then disturbed in a manner such that a pulse is created, and this pulse then travels to the right, a distance over some period of time as can be in Figure 1:

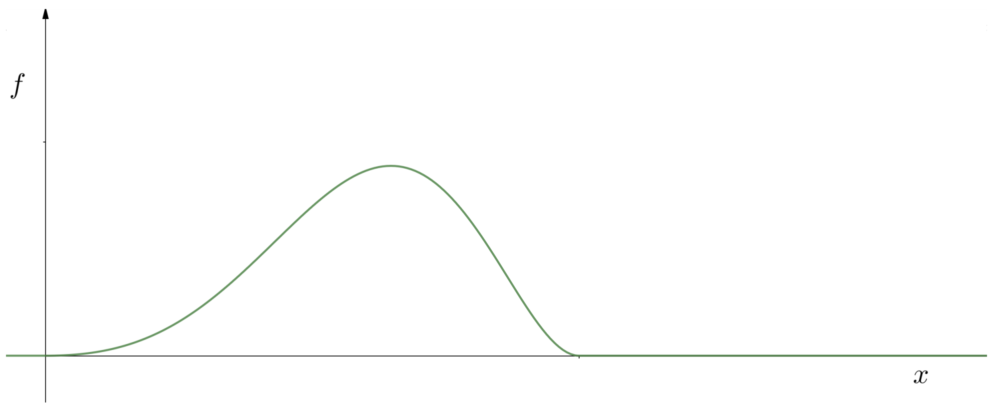

Figure 1: Pulse at time $t = 0$

When this pulse that is created, it has two properties that it obeys by throughout its existence. It does not lose its original shape and that all parts of the pulse move at equal speeds.

Acting on this geometry of the propagating pulse, we can create a function f, dependent on the two changing variables, space, x, and time, t. Which we can write as $f(x,t)$, where $f(x,t)$ is a multivariate function, in that it takes two inputs, defined previously, and for which the operations of differentiation in the form of partial differentiation and integration holds.

Definition 2.1. Hence, we can define the shape, $f(x,t)$, of the pulse by:

$$f(x,t) = \begin{cases} \phi(x) & t = 0 \\ \phi(x - ct) & t > 0 \end{cases}$$

where the function, $\phi(x)$ is simply $f(x,t)$ evaluated at $t = 0$.

The argument of $x - ct$ comes from the fact that the pulse moves at a constant speed, hence by the definition of speed, $c = \frac{x}{t}$ which yields, when solving for x, $x = ct$. Which also applies for any point on the string.

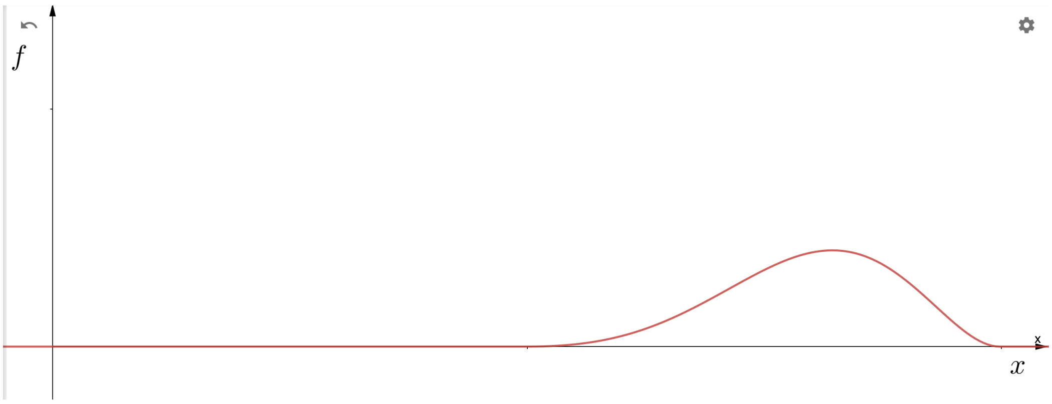

Figure 2: Pulse at time $t \geq 0$

Hence from this we can assume that the equation of which we are in the pursuit of has as it's solution $\phi(x - ct)$ for any function $\phi(x)$.

Here we make the omission of solutions for waves travelling in the opposite direction, as [4] points out, in which case we may add on another arbitrary spacial function the same as $\phi(x)$, only with it's shift being of the nature $x + ct$, which follows from the argument made for $\phi(x)$. We can define this new arbitrary function as $\psi(x + ct)$ as can be seen in Figure 2 as such:

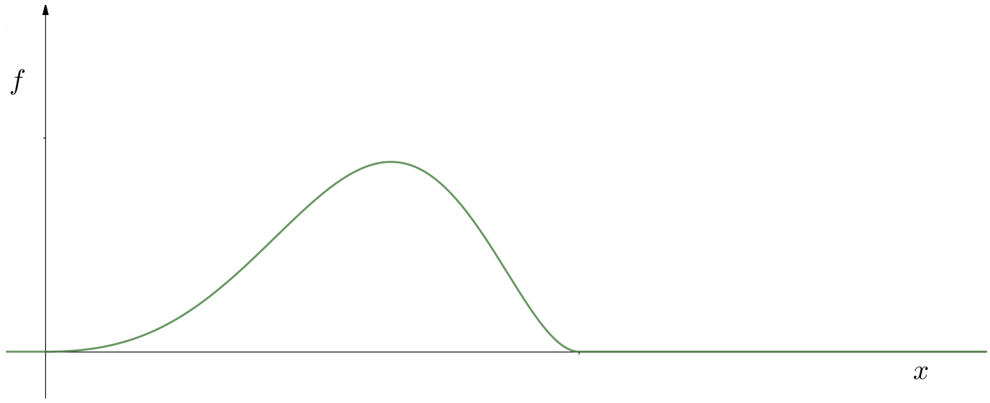

Figure 3: Pulse at time $t \geq 0$

Now, having the general solution to our equation we only have the task of finding the equation itself. We can approach this final step by the use of calculus, and partial differentiation.

Definition 2.2. Partial Differentiation: The partial derivative, much like the ordinary derivative, can be defined in terms of a limit as such, as shown by Nykamp[5]:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

To make notation more clear, derivatives with respect to time will be indicated with dots,

$$\frac{\partial f}{\partial t} = \dot{f}$$

and derivatives with respect to space, x, will be indicated with dashes,

$$\frac{\partial f}{\partial x} = f'.$$

Differentiating our function f with respect to our variables x and t could be a start.

$$\frac{\partial f}{\partial x} = \phi'(x - ct), \qquad \frac{\partial f}{\partial t} = -c\phi'(x - ct), \tag{1}$$

We can then set up a relationship of the form:

$$\frac{\partial f}{\partial x} = -\frac{1}{c}\frac{\partial f}{\partial t}.$$

Applying the same logic to $\psi(x + ct)$ as applied to $\phi(x - t)$, we can set up an equation of the form:

$$\frac{\partial f}{\partial x} = \frac{1}{c}\frac{\partial f}{\partial t}.$$

Since we are looking for two solutions, this simple first order relation will not be sufficient. So we attempt to differentiate once more, [4].

$$\frac{\partial^2 f}{\partial x^2} = \phi''(x - ct), \qquad \frac{\partial^2 f}{\partial t^2} = -c^2\phi''(x - ct),$$

Now rearranging,

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

Hence we have an equation, that once solved, will provide a general solution in the form of a linear sum of an arbitrary function, $\phi(x - ct)$ and $\psi(x + ct)$.

3 Solving the Wave Equation

3.1 Separation of Variables with Boundary and Initial Conditions

To begin crafting a solution, a method of separation of variables [6] can be applied, which will make the task significantly easier. This method helps us due to the fact that the wave function we are attempting to solve for is a multivariable function, hence separating it into its respective components could perhaps aid in our efforts.

For the sake of clarity, let us once again state the derived wave equation.

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \quad \forall\, 0 < x < l, t > 0. \tag{2}$$

We can assign boundary and initial conditions for the wave function which we are in the search of. Thus, we can write these conditions as such:

Boundary Conditions:

$$f(0, t) = 0 \quad \forall\, t > 0, \tag{3}$$

$$f(l, t) = 0 \quad \forall\, t > 0, \tag{4}$$

Initial Conditions:

$$f(x,0) = g(x) \quad \forall\, 0 < x < l, \tag{5}$$

$$\frac{\partial f}{\partial x}|_{t=0} = h(x) \quad \forall\, 0 < x < l. \tag{6}$$

It is worth to mention that the function $g(x)$ describes the spatial portion of the wave function only at $t = 0$. This is also true for the function of $h(x)$, as it describes the speed of the wave function at $t = 0$. This is since the partial derivative with respect to x of f yields speed.

Now that we have our boundary conditions, the separation of variables can be employed simply as such [6]:

$$f(x,t) = A(x)T(t).$$

Here we find that the solutions to (2) are a product of two functions, some function $A(x)$ that is dependent on x, and another function $T(t)$, which is dependent on t. The exact details on why this method of separation of variables works would require analysis out of the scope of this investigation to even attempt to explain. The exact specifications can be found to be given by Young [6].

Here we can employ what is called the principle of superposition, as given by Johnson [7].

Definition 3.1. If f_1 is a solution to a second order, linear, PDE, then so is

$$\sum_{i=1}^{n} f_i$$

But how do we know that the principle of superposition works for (2), and what does it mean to be linear?

Theorem 1. *If $f_1, f_2, ..., f_n$ are solutions to the second order, linear, PDE $f = \ddot{f} - c^2 f'' = 0$, then so is the linear combination $f = c_1 f_1 + c_2 f_2 + ... + c_n f_n$ for any c_j 's $\in \mathbb{R}$ [7].*

Proof. For a PDE to be linear, we say that it must satisfy $F(v + w) = F(v) + F(w)$ and $F(av) = aF(v)$ for some functions v and w, and some constant a, where F is of the form in

$F(x,t) = f'' - c^2\ddot{f} = 0$. Therefore, we can show $F(v+w) = F(v) + F(w)$ by,

$$\begin{aligned}
F(v+w) &= \frac{\partial^2}{\partial t^2}(v+w) - c^2\frac{\partial^2}{\partial x^2}(v+w), \\
&= \ddot{v} + \ddot{w} - c^2(v'' + w''), \\
&= \ddot{v} - c^2 v'' + \ddot{w} - c^2 w'', \\
&= F(v) + F(w).
\end{aligned} \tag{7}$$

We can also show $F(av) = aF(v)$ by,

$$\begin{aligned}
F(av) &= \frac{\partial^2}{\partial t^2}v - c^2\frac{\partial^2}{\partial x^2}v \\
&= a\ddot{v} - c^2 a v'' \\
&= a(\ddot{v} - c^2 v'') \\
&= aF(v)
\end{aligned} \tag{8}$$

Hence by (7) and (8) the PDE (2) is linear, and therefore the principle of superposition is valid.

As a result, we can make use of the principle of superposition for the wave equation, which will aid greatly toward obtaining some sort of description of the wave function we are in the search of.

Hence we can then proceed to say that: $f(x,t) = A(x)T(t)$ is only a solution to (2) if:

$$\frac{\partial^2 f}{\partial t^2} = A\frac{\partial^2 T}{\partial t^2}, \qquad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 A}{\partial x^2}T,$$

$$\Rightarrow \frac{\partial^2 A}{\partial x^2}T = \frac{1}{c^2}A\frac{\partial^2 T}{\partial t^2}.$$

Therefore (2) becomes,

$$c^2 A'' T = A\ddot{T} \iff \frac{A''}{A} = \frac{1}{c^2}\frac{\ddot{T}}{T}.$$

It should be noticed that the RHS is independent of x, therefore the LHS should as a result be independent of x. The same logic can be followed with the variable of t. This then leads us to the conclusion that both sides must be constant. Let this constant be k. Hence, by setting both sides of the equation equal to k, and rearranging, we then obtain,

$$A'' - kA = 0, \qquad \ddot{T} - c^2 kT = 0. \tag{9}$$

Therefore with this process we have in our hands now two ODEs and we can employ our ordinary techniques for solving.

We can guess a solution to both ODEs in (9) to be in the form of $A(x) = e^{c_1 x}$ and $T(t) = e^{c_2 t}$ for some constants c_1 and c_2. We do this namely due to the reason that as we know from calculus,

$$\frac{d}{dx}e^x = e^x$$

we generally obtain the desire result when substituting this in to an ODE.

Hence when we substitute our guesses in, we obtain that:

$$c_1 = \pm\sqrt{k},$$

$$c_2 = \pm c\sqrt{k}.$$

To progress, we can play around with our separation constant, in that we can consider two cases, $k = 0$ and $k \neq 0$. Playing around with these cases will ultimately allow us to arrive at some type of solution as we essentially consider all possible values of $k \in \mathbb{R}$.

For $k = 0$, the ODEs in (9) simplify quite nicely to

$$A'' = 0 \quad \ddot{T} = 0,$$

which lead to the general solutions of

$$A(x) = \gamma_1 + \gamma_2 x,$$

$$T(t) = \gamma_3 + \gamma_4 t.$$

For $k \neq 0$, we obtain two solutions for $A(x)$ and two solutions for $T(t)$,

$$A(x) = \gamma_1 e^{\sqrt{k}x} + \gamma_2 e^{-\sqrt{k}x},$$

$$T(t) = \gamma_3 e^{c\sqrt{k}t} + \gamma_4 e^{-c\sqrt{k}t}.$$

for some constants γ_1,γ_2,γ_3 and γ_4.

By our understanding of a possible solution therefore the wave function can take the forms of:

$$\begin{aligned} f(x,t) &= (\gamma_1 e^{\sqrt{k}x} + \gamma_2 e^{-\sqrt{k}x})(\gamma_3 e^{c\sqrt{k}t} + \gamma_4 e^{-c\sqrt{k}t}) \ \ , \ for \ k \neq 0, \\ f(x,t) &= (\gamma_1 + \gamma_2 x)(\gamma_3 + \gamma_4 t) \ \ , \ for \ k = 0. \end{aligned} \tag{10}$$

3.2 Imposition of Boundary and Initial Conditions

We wish to narrow down our possible answer pool, and to do this we should see which of the two cases we have, for the possible values of the separation constant, yield trivial answers. To do this, let us first consider the boundary conditions (2) and (3). We will merely substitute $x = 0$ and $x = l$ into $A(x)$ for $k = 0$ and $k \neq 0$.

Therefore, for $k = 0$, substituting in $x = 0$, we obtain that $A(0) = \gamma_1 + \gamma_2(0) = 0 \Rightarrow \gamma_1 = 0$. substituting in $x = l$, $A(l) = \gamma_1 + \gamma_2(l) = 0 \Rightarrow \gamma_1 = -\gamma_2 l$. And so from, this, we can see that our boundary condition of $A(0) = A(l) = 0$ is only satisfied if $\gamma_1 = \gamma_2 = 0 \Rightarrow A(x) = 0$. Hence the case of $k = 0$ is trivial as it produces $f(x,t) = 0$.

The focus is now on the case of $k \neq 0$, and we can apply the same process here. substituting in $x = 0$, we get that $A(0) = \gamma_1 + \gamma_2 = 0 \Rightarrow \gamma_1 = -\gamma_2$. substituting in $x = l$, $A(l) = \gamma_1 e^{\sqrt{k}l} + \gamma_2 e^{-\sqrt{k}l} = 0$. Since we know $\gamma_1 = -\gamma_2$, we can factor γ_1, $\gamma_1(e^{\sqrt{k}l} - e^{-\sqrt{k}l})$. Since $\gamma_1 \neq 0$ then $e^{\sqrt{k}l} - e^{-\sqrt{k}l} = 0$. We then obtain that $e^{2\sqrt{k}l} = 1$. This equation has an infinite number of complex solutions, hence by this we can attempt to solve for k.

$$\begin{aligned} e^{2\sqrt{k}l} &= 1, \\ \Leftrightarrow 2\sqrt{k}l &= 2n\pi i, \\ \Leftrightarrow \sqrt{k} &= \frac{n\pi i}{l}, \\ k &= -\frac{n^2\pi^2}{l^2}. \end{aligned}$$

Therefore, we can then substitute this new value of k in as such, $A(x) = \gamma_1 e^{\frac{n\pi i}{l}x} + \gamma_2 e^{-\frac{n\pi i}{l}x}$ and $T(t) = \gamma_3 e^{c\frac{n\pi i}{l}t} + \gamma_4 e^{-c\frac{n\pi i}{l}t}$. Using Euler's Formula to expand these results, it follows that,

$$A(x) = \gamma_1(2i\sin(\frac{n\pi}{l}x)),$$

$$T(t) = (\gamma_3 + \gamma_4)\cos(\frac{n\pi c}{l}t) + (\gamma_3 - \gamma_4)\sin(\frac{n\pi c}{l}t).$$

Thus, $A(x)T(t)$ becomes

$$\begin{aligned} f(x,t) = A(x)T(t) &= [\gamma_1 2i\sin(\frac{n\pi}{l}x)][\gamma_3 + \gamma_4)\cos(\frac{n\pi c}{l}t) + (\gamma_3 - \gamma_4)\sin(\frac{n\pi c}{l}t)], \\ &= \sin(\frac{n\pi}{l}x)[\gamma_1 2i(\gamma_3 + \gamma_4)\cos(\frac{n\pi c}{l}t) + \gamma_1 2i(\gamma_3 - \gamma_4)\sin(\frac{n\pi c}{l}t)], \\ &= \sin(\frac{n\pi}{l}x)[\alpha_n\cos(\frac{n\pi c}{l}t) + \beta_n\sin(\frac{n\pi c}{l}t)]. \end{aligned}$$

In the final step, we introduced α_n and β_n to become a part of the solution. This is done not only to simplify to problem, but also allow us to explore further into how manipulations done upon these coefficients can effect the nature of the wave function. Writing more explicitly, we can say that,

$$f(x,t) = \sin(\frac{n\pi}{l}x)[\alpha_n\cos(\frac{n\pi c}{l}t) + \beta_n\sin(\frac{n\pi c}{l}t)] \quad \alpha_n, \beta_n \in \mathbb{C}. \tag{11}$$

The reason as to why $\alpha_n, \beta_n \in \mathbb{C}$ will also be evident as we consider α_n and β_n individually. Proceeding, given such a solution $f(x)$, using Theorem 1, it follows that we can sum over f such that

$$f(x,t) = \sum_{n=1}^{\infty}\sin(\frac{n\pi}{l}x)[\alpha_n\cos(\frac{n\pi c}{l}t) + \beta_n\sin(\frac{n\pi c}{l}t)]. \tag{12}$$

From this point, we must also now take into consideration our initial conditions. And as tools, we can use orthogonality relations for $\sin(x)$ such that we can choose any α_n and β_n to fit our initial conditions.

Definition 3.2. The orthogonality relation of $\sin(x)$ is defined by Strang [8] as such:

$$\int_0^l \sin(nx)\cdot\sin(mx)dx = \begin{cases} \frac{l}{2} & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

From initial condition (5),

$$f(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin(\frac{n\pi}{l}x) = g(x),$$

We can now exploit orthogonality relationship of $\sin(x)$ Multiplying by $\sin(\frac{n\pi}{l}x)$ and integrating over the length of string $(0, l)$, defined in the initial conditions, on both sides [9], we obtain,

$$\alpha_n \int_0^l \sin^2(\frac{n\pi}{l}x)dx = \int_0^l g(x)\sin(\frac{n\pi}{l}x)dx.$$

It is easy then to see that the LHS of this equation simply becomes $\alpha_n \frac{l}{2}$ which implies that,

$$\alpha_n = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi}{l}x)dx. \tag{13}$$

Similar to (13), it also follows from initial condition (6), that

$$\dot{f}(x,0) = \sum_{n=1}^{\infty} \beta_n \frac{n\pi c}{l}\sin(\frac{n\pi}{l}x) = h(x)$$

From the same argument made for α_n, we obtain

$$\beta_n \int_0^l \sin^2(\frac{n\pi}{l}x)dx = \int_0^l h(x)\sin(\frac{n\pi}{l}x)dx,$$

which implies that,

$$\beta_n = \frac{2}{n\pi c}\int_0^l h(x)\sin(\frac{n\pi}{l}x)dx. \tag{14}$$

Now that we have applied our initial and boundary conditions to the possible solutions in (11), we can say that the solution to (2) is in the form,

$$f(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{l}x)[\alpha_n \cos(\frac{n\pi c}{l}t) + \beta_n \sin(\frac{n\pi c}{l}t)] \tag{15}$$

where α_n, β_n are given by (13) and (14).

Infact, solutions of this form, sums of $\sin(x)$, $\cos(x)$, are called Fourier Series [10], named after Joseph Fourier, who first investigated these sums. Although this name won't be used in this paper, this is for purposes of identification.

3.3 Uniqueness and Existence of General Solution

An important question now comes up, in that how do we know that this solution we have found is the only solution that satisfies the PDE with the specific boundary conditions? Further still, how can we even tell if this solution exists? How can we verify the existence and uniqueness for (15)?

Using a similar energy argument to Amaranth [11], we may argue the following:

Theorem 2. *Given the following PDE with boundary conditions, if there is a solution, it is unique*

$$\begin{aligned} \ddot{f} &= c^2 f'' \quad \forall\, 0 < x < l, t > 0 \\ f(0,t) &= f(l,t) = 0, \quad \forall\, t > 0, \\ f(x,0) &= 0, \quad \forall\, 0 < x < l, \\ f'(x,0) &= 0, \quad \forall\, 0 < x < l \end{aligned}$$

Proof. Assume f_1 and f_2 are two solutions. We must show that $f_1 = f_2$. Consider then a $w = w(x,t)$ such that $w = f_1 - f_2$, and that w satisfies the PDE in (2) such that,

$$\ddot{w} = c^2 w'' \Leftrightarrow \ddot{w} - c^2 w'' = 0$$

Given the following conditions,

$$\begin{aligned} w(0,t) &= w(l,t) = 0, \quad \forall t > 0 \\ w(x,0) &= 0 \\ \dot{w}(x,0) &= 0. \end{aligned}$$

So if the conditions above are met, then we must show $w(x,t) \equiv 0$ for $0 < x < l, t > 0 \Rightarrow f_1(x,t) = f_2(x,t)$. Hence, given $w(x,t)$ which satisfies (2) we can define the total energy of the string or wave to be:

$$E(t) = \frac{1}{2} \int_0^l (\dot{w}^2 + c^2 w'^2) dx$$

$E(t)$ is simply a sum of the kinetic, $\dot{w}^2$ and potential, $c^2 w'^2$, energies of the system defined previously with respect to w. However, since this is the total energy of the system, as it was assumed to contain no energy losses, it must follow conservation of energy, and thus

$$\dot{E} = 0 \tag{16}$$

So if $\dot{E} = 0$, $\Rightarrow E(t) = constant$. So by computing $E(0)$,

$$E(0) = \frac{1}{2}\int_0^l (\dot{w}^2 + c^2 w'^2)dx = \frac{1}{2}\int_0^l (0+0)dx = 0 \ \ \forall t > 0$$

Meaning that

$$E(t) = \frac{1}{2}\int_0^l (\dot{w}^2 + c^2 w'^2)dx \equiv 0 \tag{17}$$

And so for $E(t) = 0$ then $\frac{1}{2}(\dot{w}^2 + c^2 w'^2) \geq 0$. However, if $\frac{1}{2}(\dot{w}^2 + c^2 w'^2) \neq 0$, then it is not possible for $E(t) = 0$. Hence the only way for this to be true is if $\frac{1}{2}(\dot{w}^2 + c^2 w'^2) = 0$, thus $\Rightarrow w'(x,t) = 0, \dot{w}(x,t) = 0 \Rightarrow w(x,t) = 0 \Rightarrow f_1 = f_2$. Hence, uniqueness.

The solution that we have found in (15) is therefore indeed a unique solution within the boundary and initial conditions that we have provided for it, and can procede to analyze it in the context of music.

4 Manipulation of Vibrating Strings

4.1 Decay of Harmonics

To investigate the nature of α_n and β_n, we attempt to investigate an important result in analysis. As was stated in equation 15, α_n and β_n are some constants which control the magnitude of the movement of the string. These constants are called harmonics, and hold discrete values. It is also true for these harmonics that α_1 has the greatest value out of all harmonics. But is this always the case? To answer this question, the following Theorem, which is a variation on Parseval's Theorem [12], must be considered.

Theorem 3. *Given a function which has a Fourier Series approximation of*

$$f(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{l}x)[\alpha_n \cos(\frac{n\pi c}{l}t) + \beta_n \sin(\frac{n\pi c}{l}t)]$$

then the following holds true

$$\frac{l^2}{c} \iint_{-\pi}^{\pi} f(x,t)^2 \, dx \, dt = \sum_{n=1}^{\infty} \alpha_n^2 + \beta_n^2$$

Given that $f(x,t)$ is integrable on $[-\pi, \pi]$.

Proof. It is given that $f(x,t)$ can be written of the form

$$f(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{l}x)[\alpha_n \cos(\frac{n\pi c}{l}t) + \beta_n \sin(\frac{n\pi c}{l}t)].$$

We can perform a process of now squaring and integrating f with respect to x and t to observe a resulting integral-sum pair. Writing $\frac{n\pi}{l} = \lambda_n$,

$$\iint_{-\pi}^{\pi} (f(x,t))^2 \, dx \, dt = \iint_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \sin(\lambda_n x)[\alpha_n \cos(\lambda_n ct) + \beta_n \sin(\lambda_n ct)] \right)^2 dx \, dt$$

Which necessarily implies that

$$\iint_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \sin(\lambda_n x)\alpha_n \cos(\lambda_n ct) + \sum_{n=1}^{\infty} \sin(\lambda_n x)\beta_n \sin(\lambda_n ct) \right)^2 dx \, dt$$

$$= \iint_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \sin(\lambda_n x)\alpha_n \cos(\lambda_n ct) \right)^2 + 2 \left(\sum_{n=1}^{\infty} \sin(\lambda_n x)\alpha_n \cos(\lambda_n ct) \cdot \sum_{n=1}^{\infty} \sin(\lambda_n x)\beta_n \sin(\lambda_n ct) \right) + \left(\sum_{n=1}^{\infty} \sin(\lambda_n x)\beta_n \sin(\lambda_n ct) \right)^2 dx \, dt$$

And hence due to the linearity of the integral, the main double integral can be segmented as such.

$$I_1 = \iint_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \sin(\lambda_n x)\alpha_n \cos(\lambda_n ct) \right)^2 dx \, dt$$
$$I_2 = \iint_{-\pi}^{\pi} 2 \left(\sum_{n=1}^{\infty} \sin^2(\lambda_n x)\alpha_n \beta_n \cos(\lambda_n t) \sin(\lambda_n t) \right) dx \, dt$$
$$I_3 = \iint_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \sin(\lambda_n x)\beta_n \sin(\lambda_n ct) \right)^2 dx \, dt$$

It is important here to define and introduce some important concepts so as the steps to be taken are well understood.

Definition 4.1. The Kronecker-delta function, δ_{mn} as defined by [13] is a function of two variables, m and n such that if $n = m$ then $\delta_{mn} = 1$ and if $m \neq n$, $\delta_{mn} = 0$. More specifically,

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

As a concrete example, by the definition of the kronecker-delta, we have that $\delta_{12} = \delta_{23} = \delta_{34} = 0$ while $\delta_{11} = \delta_{22} = \delta_{33} = 1$. This idea can be readily applied to sums, as will be presented in an example. Given a sum,

$$\sum_{n=1}^{3} \alpha_n = \alpha_1 + \alpha_2 + \alpha_3 = 4 + 5 + 6 = 15$$

We can place a Kronecker-delta with indexed n, m as $n, 2$, and we would subsequently obtain,

$$\sum_{n=1}^{3} \alpha_n \delta_{n2} = \alpha_1 \delta_{12} + \alpha_2 \delta_{22} + \alpha_3 \delta_{32} = 4 \cdot 0 + 5 \cdot 1 + 6 \cdot 0 = 5$$

Another property which will be specifically of use within this proof is that of the fact that, if given a sum running from 1 till some $N \in \mathbb{Z}^+$ then

$$\sum_{n=1}^{N} \alpha_n \beta_n = \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \beta_m \delta_{nm} \tag{18}$$

Hence given this we can begin to calculate the double integral I_1 as such

$$\begin{aligned} I_1 &= \iint_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \sin(\lambda_n x) \alpha_n \cos(\lambda_n ct) \right)^2 dx\, dt \\ &= \iint_{-\pi}^{\pi} \sum_{n=1}^{\infty} \sin(\lambda_n x) \alpha_n \cos(\lambda_n ct) \cdot \sum_{m=1}^{\infty} \sin(\lambda_m x) \alpha_m \cos(\lambda_m ct)\, dx\, dt \end{aligned}$$

The splitting of the square of a singular series can be done, so as the fact that $n = m$ is kept in mind, especially during integration, as the Kronecker-delta makes this possible. From this step, the sums and integrals can be interchanges to produce

$$\begin{aligned}
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_n \alpha_m \iint_{-\pi}^{\pi} \sin(\lambda_n x)\cos(\lambda_n ct)\cdot \sin(\lambda_m x)\cos(\lambda_m ct)\,dx\,dt \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_n \alpha_m \int_{-\pi}^{\pi} \sin(\lambda_n x)\sin(\lambda_m x) \int_{-\pi}^{\pi} \cos(\lambda_m ct)\cos(\lambda_n ct)\,dx\,dt \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_n \alpha_m \cdot \delta_{nm}\frac{c}{l^2}
\end{aligned}$$

And hence, we can make use of equation (18) by setting $\alpha_n = \beta_m$, and therefore obtaining final result that

$$I_1 = \frac{c}{l^2}\sum_{n=1}^{\infty}\alpha_n^2$$

By repeating the same process, it would be the case that

$$I_3 = \frac{c}{l^2}\sum_{n=1}^{\infty}\beta_n^2$$

Hence, by combining we would have

$$\iint_{-\pi}^{\pi}(f(x,t))^2\,dx\,dt = \frac{c}{l^2}\sum_{n=1}^{\infty}\alpha_n^2 + \iint_{-\pi}^{\pi} 2\left(\sum_{n=1}^{\infty}\sin^2(\lambda_n x)\alpha_n\beta_n\cos(\lambda_n t)\sin(\lambda_n t)\right)\,dx\,dt + \frac{c}{l^2}\sum_{n=1}^{\infty}\beta_n^2$$

and by observing that

$$\iint_{-\pi}^{\pi} 2\left(\sum_{n=1}^{\infty}\sin^2(\lambda_n x)\alpha_n\beta_n\cos(\lambda_n t)\sin(\lambda_n t)\right) = 0$$

by orthogonality relations of $\sin(x)$ defined previously, and thus by rearranging, we see that

$$\frac{l^2}{c}\iint_{-\pi}^{\pi} f(x,t)^2\,dx\,dt = \sum_{n=1}^{\infty}\alpha_n^2 + \beta_n^2$$

This statement is quite remarkable, as it is essentially stating that the sums of the squares of the Fourier series coefficients of some function $f(x)$ are finite. More clearly, because $f(x)$ is integrable on $[-\pi, \pi]$, then it has a finite value, which means that the series on the right hand side is also subsequently finite.

Definition 4.2. Absolute convergence of an infinite series is if an infinite series, $\sum_{n=0}^{\infty} a_n$

converges to some real number L, i.e, if

$$\sum_{n=0}^{\infty} \mid a_n \mid = L$$

as was defined by Bromwich and MacRobert [14].

This definition also implies that perhaps changing around values or terms within the sum, rearranging terms more specifically do not yield a different value, which means no amount of trickery within the terms of the series will yield anything different.

$$\sum_{n=1}^{\infty} \alpha_n^2 + \beta_n^2 = \sum_{n=1}^{\infty} \alpha_n^2 + \sum_{n=1}^{\infty} \beta_n^2$$

Given this, since it is known that these series must converge, the nature of this convergence can be taken into account.

$$\sum_{n=1}^{\infty} \alpha_n^2 \approx \sum_{n=1}^{\infty} \mid \alpha_n \mid = L$$

Observe further that this series contains a sequence, α_n.

Corollary 3.1. If the series $\sum_{n=1}^{\infty} \mid \alpha_n \mid = L$, then the sequence α_n has a limit $\lim_{n\to\infty} \alpha_n = 0$

Proof. Since its known that the series must converge, then the sequence α_n must subsequently converge to 0. Namely, we can consider the partial sums α_n

$$S_n = \sum_{k=1}^{n} \alpha_k$$

$$\Rightarrow \alpha_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$\Rightarrow \lim_{n\to\infty} \alpha_n = 0$$

This necessarily means that the sequence of harmonics α_n, by Theorem 3 and Corollary 3.1, must decay as $n \to \infty$.

5 Connections to Music Theory

If a composer is constructing a piece of music, the instruments should be able to be played in such as way so as to provide the sound the composer is looking for. In stringed instruments, it would be possible to pluck the string of the instruments within specific spots [15]. Mathematically, this pluck can be formulated by the manipulation of an initial-shape function, $g(x)$ as proposed in Section 3.2. For some string with length $x = l$, starting from some position which will be $x = 0$, the following initial-shape function can be proposed $g(x) = x(l - x)$.

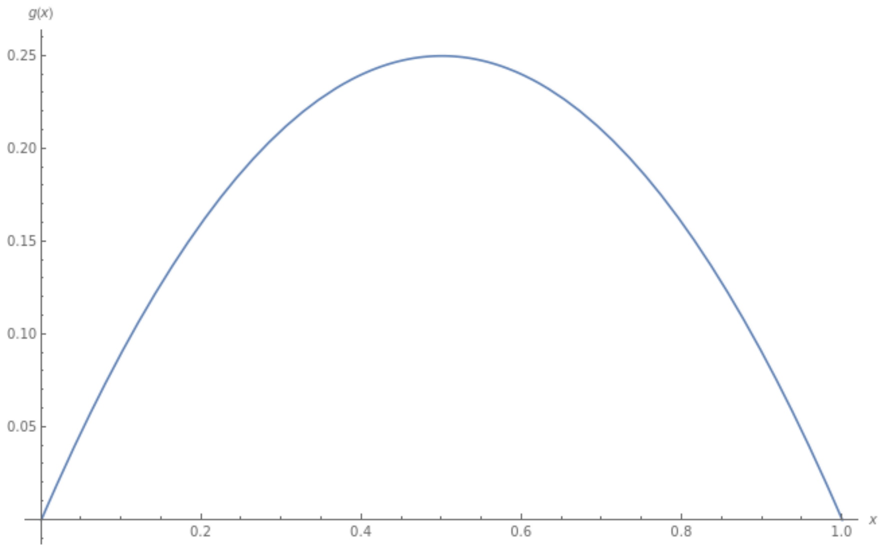

Figure 4: Initial-shape Function $g(x)$ with $l = 1$

Although this shape isn't indicative of all possible shapes of strings produced by a player, we may introduce a plucking position, k so as both to generalize this shape and connect to the production of sounds by altering plucking position. Thus we re-define $g(x)$ as piece-wise function:

$$g(x) = \begin{cases} kx(l-x) & 0 \le x \le k \\ -k^2(x-l) & k < x \le l \end{cases}$$

with k ranging from $0 < k < l$ being the specific position at which the string is plucked.

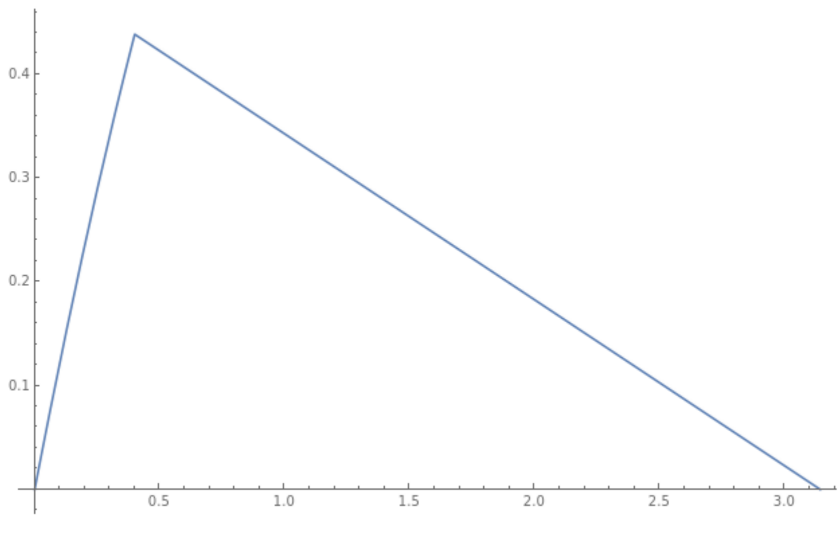

Figure 5: Initial-shape Function $g(x)$ with $l = \pi$ and $k = 0.2$

It should be noted here that choosing $l = \pi$ was not an arbitrary decision, and was chosen so as to limit the burden of calculations which will be showcased further on. Choosing a specific value of l also helps in observing one family of various functions by simply varying k. Going back to the question of how exactly this influences the sound which emanates from the string, reference can be made to Equation 13. Due to $g(x)$ being of the form of a piece-wise function, the integral may be split up into two distinct integrals:

$$\alpha_n = \frac{2}{\pi}\left(\int_0^k kx(\pi - x)\sin(nx)dx + \int_k^\pi (-k^2)(x-\pi)\sin(nx)dx\right)$$

Using a process of repeated integration by parts as such

$$\begin{aligned} I_1 + I_2 &= \int_0^k kx(\pi - x)\sin(nx)dx + \int_k^\pi (-k^2)(x-\pi)\sin(nx)dx \\ &= k\int_0^k x(\pi - x)\sin(nx)dx + k^2\int_k^\pi (x-\pi)\sin(nx)dx \end{aligned}$$

Now considering I_1:

$$\begin{aligned} I_1 &= k\int_0^k x(\pi - x)\sin(nx)dx \\ &= k\int_0^k x\pi\sin(nx) - x^2\sin(nx)dx \\ &= k\left(\int_0^k x\pi\sin(nx)dx - \int_0^k x^2\sin(nx)dx\right) \\ &= k\left(\frac{\pi}{n^2}\sin(kn) - \frac{k\pi}{n}\cos(kn) - \frac{1}{n}k^2\cos(nk) + \frac{2}{n^3}\left(nk\sin(nk) + \cos(nk)\right) - \frac{2}{n^3}\right) \\ &= \frac{k(-2 + (2 + kn^2(-k+\pi))\cos(kn) - n(-2k+\pi)\sin(kn)}{n^3}, \end{aligned}$$

and similarly I_2:

$$\begin{aligned} I_2 &= k^2\int_k^\pi (x-\pi)\sin(nx)dx \\ &= k^2\int_k^\pi x\sin(nx) - \pi\sin(nx)dx \\ &= k^2\left(\int_k^\pi x\sin(nx)dx - \int_k^\pi \pi\sin(nx)dx\right) \\ &= k^2\left(-\frac{1}{n}\left(\frac{1}{n}\sin(nk) - \cos(nk)(k-\pi)\right)\right) \\ &= \frac{k^2(-n(-k+\pi)\cos(kn) - \sin(kn) + \sin(n\pi))}{n^2}. \end{aligned}$$

Thus, yielding in the end

$$\begin{aligned}\alpha_n =& \frac{2}{\pi}\frac{k(-2+(2+kn^2(-k+\pi))\cos(kn)-n(-2k+\pi)\sin(kn)}{n^3} \\ &+\frac{2}{\pi}\frac{k^2(-n(-k+\pi)\cos(kn)-\sin(kn)+\sin(n\pi))}{n^2}.\end{aligned} \tag{19}$$

Given this expression for α_n, the strengths of individual harmonics can be examined, to give perhaps a holistic view of the overall sound produced at a single point. The following is a plot of the amplitude of harmonics $|\alpha_n|$ as a function of plucking position, k

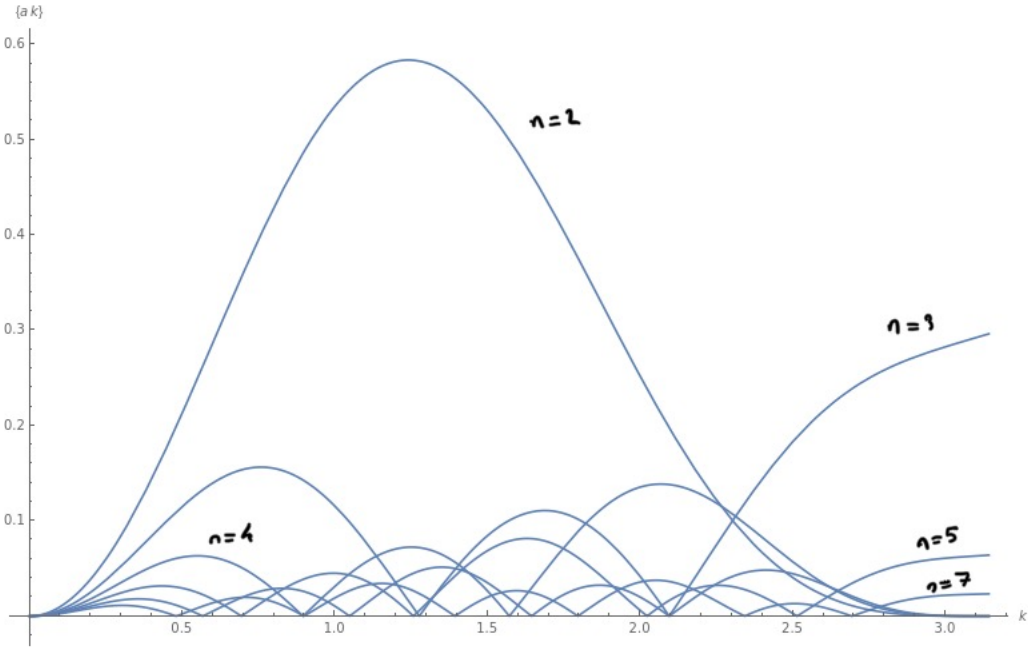

Figure 6: Initial-shape Function $g(x)$ with $n = 2, 3, 4, 5, 6, 7$

Thus we see that in Figure 6,

$$\lim_{k\to l} \alpha_n(k) \neq 0 \quad \forall n = 2m+1,\ m \in (\mathbb{Z}^+ \cup \{0\}) \tag{20}$$

$$\lim_{k\to l} \alpha_n(k) = 0 \quad \forall n = 2m,\ m \in (\mathbb{Z}^+ \cup \{0\}) \tag{21}$$

The main take-away from this observation is that if the string is plucked in a certain position, if that position contains the zeroes of a multitude of harmonics, those harmonics will simply disappear. In general, it can be conjectured that given the above system, it will be the case that $\alpha_n \neq 0$, when $n = 2m+1$ and similarly, $\alpha_n = 0$, when $n = 2m$ for some $m \in \mathbb{N}$. This does not contradict

$$\lim_{n\to\infty} \alpha_n = 0,$$

since this is when n approaches infinity, and for some finite $n \in \mathbb{N}$, the above would be plausible.

5.1 The Production of Sound by Vibrating Strings

Let us first consider the initial movement of the string, what happens to the string after a time $t = 1$? Taking a look at $\sin(x)$ and $\cos(x)$ terms containing t, it can be observed that their arguments become $\frac{n\pi c}{l}$ which means that the period observed upon these terms is $\frac{nc}{2l}$. Meaning that the discrete frequencies at which the string oscillates, or harmonics, are of the form

$$f_n = \frac{nc}{2l}, \quad n \in \mathbb{Z}^+. \tag{22}$$

Carrying values used previously, a sample calculation can be performed, such that if $l = \pi$ and $c = 1$, then the sequence of frequencies is as such $f_1 = \frac{1}{2\pi}$,$f_2 = \frac{1}{\pi}$,$f_3 = \frac{3}{2\pi}$. f_1 can be denoted as the *fundamental* frequency, in that its the lowest frequency which is a part of the motion of the string. Frequencies above the fundamental are denoted as *overtones*. The following is an example of overtones in musical notation, given by [16], which is essentially what is contained within our solution in Equation 19!

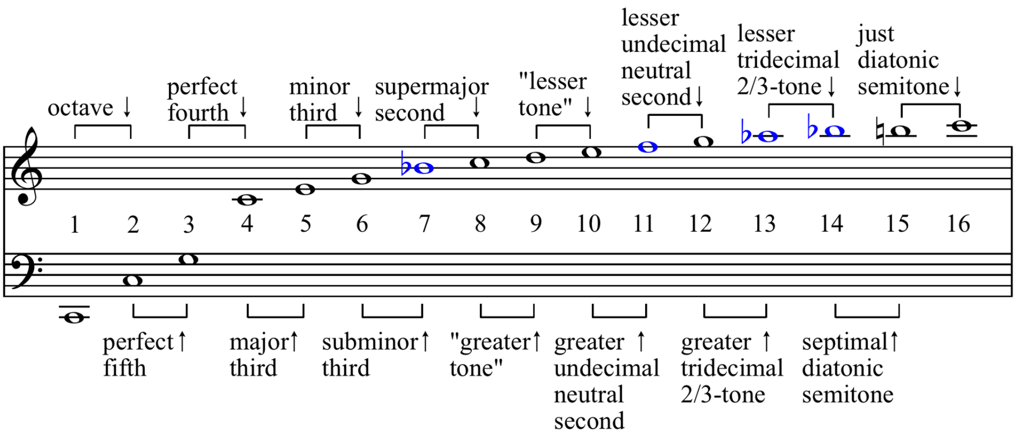

Figure 7: Overtones in Musical Notation

Thus, it is true within the excitation of an instrument to produce some type of sound, a note, that that note is a composition of finitely many overtones and a single fundamental. The fundamental itself has to be present in every single note played by that instrument, and the overtones which follow must only add on a slight amount to this fundamental. Using the example previously, the extent to which this statement is true can be seen as such:

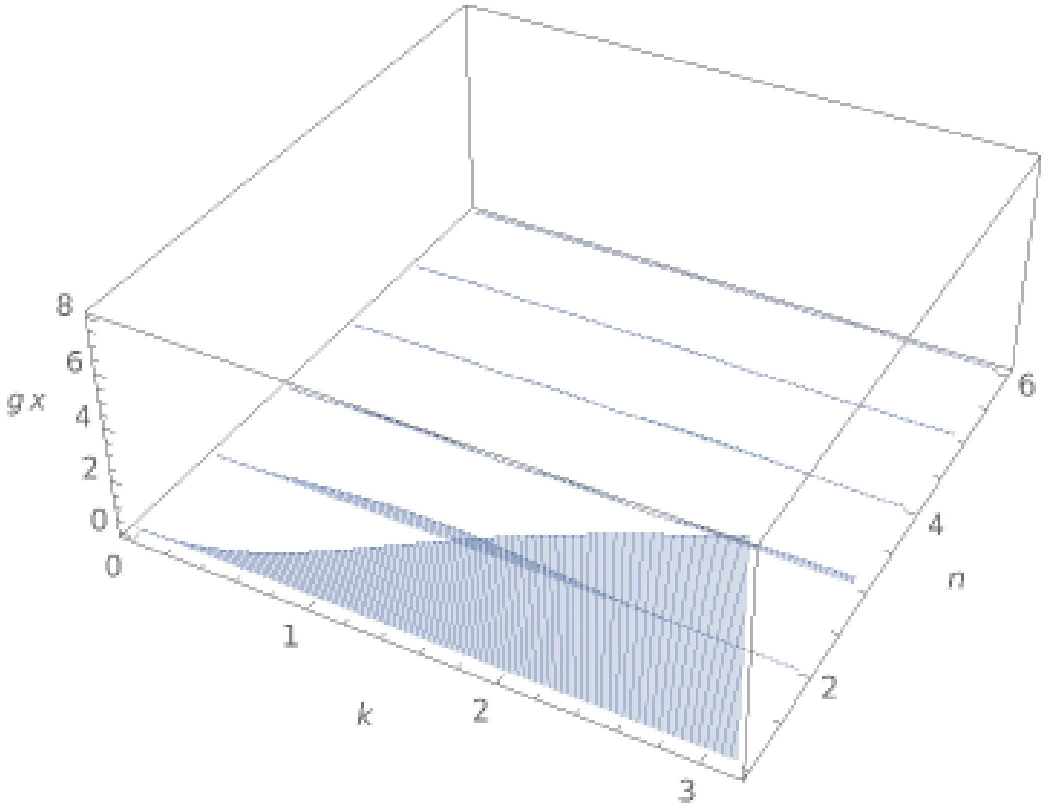

Figure 8: Amplitudes of Harmonics as a Function of k at Discrete n

In Figure 8, it can be observed that the fundamental dominates over the subsequent harmonics, and the overtones only add a tiny fraction to the overall sound produced. Hence, in true musical instruments higher-order overtones tend to contribute only a little, and they might as well just not be there at all. One may conjecture that only overtones of order $n = 5$ that are usually of concern. Subsequently, the combination of these few overtones and fundamental, which can be denoted in sum as *partials*, is what matters in analyzing the sound produced.

5.2 Consonance and Dissonance

Although the terms of consonance and dissonance may tend to have quite subjective meanings, depending on the individual who is experiencing the sound, one sound may sound quite nice, while another may sound rough and chaotic, and it might be quite the opposite for another individual. For this reason, consonance and dissonance within the context of this essay will be defined as a "mellow", and "rough" sound respectively. To be more specific, consonance and dissonance can be discussed in the range of either a single, or many notes in combination. Referring back to analysis done previously, a single note can be a dissonant one by simply the position of plucking.

Within the specific example, dissonant or "rough" sounds would be created mostly as the plucking position approaches the end of the string. This is due to the fact that there are more audible partials, all interfering with each other to create a 'rough' sound. The opposite is true for plucking at a position where most of the partials have a zero, as there would be less audible partials, hence less interference, hence a more "mellow" tone. This phenomena could be explained by grouping partials in specific sets. Let the set $\mathbb{S}$, indicated by brackets, $\{\}$ be defined as

$$\mathbb{S} = \{\alpha_n(k)\},$$

and the set $\mathbb{M}$ is $\mathbb{M} = \mathbb{S} \setminus \{\{0\}\}$, i.e., consider only the set of partials which are not null.

Definition 5.1. The cardinality of a set, as defined by Pauli [17]: Given a finite set A, such that $\mathbb{A} = \{\}$, the cardinality of A is the number of elements present in that set, denoted by $|\mathbb{A}| = n$ for $n \in \mathbb{N}$.

The cardinality of this set, $|\mathbb{M}|$, gives the number of partials which are not zeroes at a position along the string k. Taking the ratio $\frac{|\mathbb{M}|}{|\mathbb{S}|}$ would give the percentage of partials which are audible at some position k, and hence to provide a consonant tone, this ratio would need to be, per rough definition,

$$\frac{|\mathbb{M}|}{|\mathbb{S}|} > 0.5. \tag{23}$$

For the case of two notes, it is the ratios of the fundamentals which is the determinant of consonance or dissonance. In general, for consonance between two notes played in unison, it should be that this ratio should be a 'simple' one, in that it is a rational or close to rational number. Some examples of these ratios are a perfect fifth, i.e $\frac{3}{2}$, a perfect fourth $\frac{4}{3}$ and a major third $\frac{5}{4}$ [18], which can all be seen in Figure 7. For two notes to be dissonant, it can be stated that pairs of overtones interfere constructively at some points, while not at others. This relationship therefore outlines how the difference within the fundamentals and subsequent overtones, as it grows bigger, the resultant sound gets more and more 'rough' [1]. To formulate all of this mathematically, given two notes with frequencies of

$$f_{n,1} = \frac{n_1 c}{2l},$$
$$f_{n,2} = \frac{n_2 c}{2l},$$

it must be true that for consonance, the notes themselves must be multiples of each other:

$$f_{n,1} = p f_{n,2}. \quad p \in \mathbb{N} \tag{24}$$

For some degree of dissonance, the differences between the frequencies must be greater than some predetermined constant, which varies for the pair of notes being considered:

$$\Delta f = (f_{n,2} - f_{n,1}) > C.$$

Hence, as Δf increases beyond the constant $C \in \mathbb{R}^+$ then the sound gets progressively more dissonant.

6 Conclusion

The investigation lead in this paper goes to show that there has been a successful exploration and investigation of the research question, managing to obtain a satisfactory, although not complete solution to the problem at hand. What stood out in the investigation was the relation of the ratio of audible harmonics, Equation (21),

$$\frac{|\mathbb{M}|}{|\mathbb{S}|} > 0.5,$$

to consonance in a single note. Or rather, the fact that a definition relating to consonance could exist in relation to a single note. This is quite interesting, as normally in many musical systems of thought, consonance is defined with relation to multiples of notes, normally 2 [1]. The reason as to why we were able to define it in relation to a single note can probably be accredited to the fact that we considered the individual harmonics which go into producing a single note, and the relations they have to each other. Although the choice of 0.5 as a bound was relatively unfounded, it could be improved perhaps via experiment.

Further, some sections, such as the choice of $g(x)$, and usage of Theorem 3, could be altered or analyzed further. Namely, the choice of $g(x)$ could be improved to reflect better the plucking of strings observed during live performances perhaps. And so therefore producing different patterns in harmonics. Theorem 3 could also be attempted to be generalized further, considering different solutions, or multi-dimensional solutions to the wave equation.

In sum however, although the extent of this analysis might have been quite limited in terms of scope and extent, it does go a long way to answer on the content which the research question poses. It also gives insight into further research questions, of the form perhaps of investigating the ideal $g(x)$ such that no matter the k, it produces the most consonant tone, or questions to that extent.

References

[1] D. Benson, *Music: A Mathematical Offering*. 2008.

[2] J. L. Davis, "The wave equation," in *Mathematics of Wave Propagation*. Princeton University Press, 2000, pp. 85–144, ISBN: 9780691026435. [Online]. Available: http://www.jstor.org/stable/j.ctv182jt69.6.

[3] M. Barile, *Uniqueness theorem*, Date Accessed: 02/01/2021. [Online]. Available: https://mathworld.wolfram.com/UniquenessTheorem.html.

[4] J. Kiskis, *Wave Equation*. University of California, 2012, Date Accessed: 21/08/2020. [Online]. Available: http://kiskis.physics.ucdavis.edu/landau/phy9hc_03/wave.pdf.

[5] D. Q. Nykamp, *Partial derivative by limit definition*, Date Accessed: 24/08/2020. [Online]. Available: https://mathinsight.org/partial_derivative_limit_definition.

[6] N. Young, "Sturm–liouville systems," in *An Introduction to Hilbert Space*. Cambridge University Press, 1988, pp. 105–118.

[7] B. Johnson, *PDE Notes*, vol. 22, Date Accessed: 14/09/2020. [Online]. Available: http://www.math.umbc.edu/~jbell/pde_notes/.

[8] G. Strang, *Computational science and engineering*. Wellesley-Cambridge Press, 2012.

[9] E. Miersemann, *Partial Differential Equations*. 2012.

[10] W. E. Byerly, "Development in trigonometric series," in *An Elementary Treatise on Fourier's Series: and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics*. Ginn amp; Company, 1893, pp. 30–31. [Online]. Available: https://archive.org/details/elemtreatfour00byerrich/page/n45/mode/2up.

[11] T. Amaranath, *An Elementary Course in Partial Differential Equations*, ser. G - Reference, Information and Interdisciplinary Subjects Series. Jones & Bartlett Learning, 2009, ISBN: 9780763762445. [Online]. Available: https://books.google.gr/books?id=uP%5C_KJzi3aPEC.

[12] E. W. Weisstein, Date Accessed: 12/12/2020. [Online]. Available: https://mathworld.wolfram.com/ParsevalsTheorem.html.

[13] D. Rollins, *Delta function*, Date Accessed: 20/12/2020, Aug. 2006. [Online]. Available: http://www.cchem.berkeley.edu/chem120a/extra/delta_functions.pdf.

[14] T. J. I. Bromwich and T. M. MacRobert, *An Introduction to the Theory of Infinite Series*, 3rd ed. American Mathematical Society.

[15] M. Campbell and C. A. Greated, *The Musician's guide to acoustics*. Oxford University Press, 2001.

[16] Hyacinth, *Harmonic series intervals*. Wikipedia, Jul. 2009, Date Accessed: 16/12/2020. [Online]. Available: https://commons.wikimedia.org/wiki/File:Harmonic_series_intervals.png.

[17] S. Pauli, S. Steward, and M. Farmer, *Definition of cardinality*, Date Accessed: 20/12/2020. [Online]. Available: https://mathstats.uncg.edu/sites/pauli/112/HTML/section-39.html.

[18] A. Visintin, *Fourier Series and Music Theory*. University of Trento, 2019, Date Accessed: 28/08/2020. [Online]. Available: https://www.science.unitn.it/~visintin/Fourier-Mus.pdf.